Analysis IV

(for EL, GM, MX)

Week 10 - 28.04.25

Let us recall from before the break:

<u>Laplace transform:</u> If $f:[0,+\infty) \longrightarrow \mathbb{C}$ is piecewise continuous,

$$\mathcal{L}[f](z) = \int_0^{+\infty} f(t)\, e^{-zt}\, dt.$$

<u>Domain of definition:</u> If $\int_0^{+\infty} |f(t)| \cdot e^{-\gamma t} dt < +\infty$ for all $\gamma > \gamma_0$, then γ_0 is called an abscissa of convergence and $\mathcal{L}[f](z)$ is holomorphic on the half plane

$$\{z : \operatorname{Re}(z) > \gamma_0\}.$$

Some explicit examples:

- $$\mathcal{L}[e^{-at}](z) = \int_0^{+\infty} e^{-at} \cdot e^{-zt}\, dt$$

$$= \frac{1}{a+z} \quad \text{for} \quad \boxed{Re(z) > -Re(a)}$$

(Valid for $a \in \mathbb{C}$).

- When $a=0$: $\mathcal{L}[1](z) = \frac{1}{z}$

Useful identities: Allow us to compute the Laplace transform of several explicit examples.

- Linearity: $\mathcal{L}[af+bg] = a\mathcal{L}[f] + b\mathcal{L}[g]$ for $a, b \in \mathbb{C}$.

- $$\frac{d}{dz}\mathcal{L}[f](z) = -\mathcal{L}[t \cdot f](z)$$

- $$\mathcal{L}[f'](z) = z \cdot \mathcal{L}[f](z) - f(0)$$

- $\mathcal{L}[f^{(n)}](z) = z^n \mathcal{L}[f] - z^{n-1} f(0) - z^{n-2} f'(0) - \ldots - f^{(n-1)}(0).$

From the above:

$$\mathcal{L}[1](z) = \frac{1}{z} \Longrightarrow \boxed{\mathcal{L}[t^n](z) = \frac{n!}{z^{n+1}}}$$

Or:

$$\mathcal{L}[t \cdot e^{-at}](z) = -\frac{d}{dz} \mathcal{L}[e^{-at}](z)$$

$$= -\frac{d}{dz}\left(\frac{1}{z+a}\right) = \frac{1}{(z+a)^2}.$$

- $\mathcal{L}\left[\int_0^t f(s)\,ds\right](z) = \frac{1}{z} \mathcal{L}[f](z)$

Since, if $g(t) = \int_0^t f(s)\,ds$, then $f = g'$ and $g(0) = 0$.

Note Even if $f \to 0$ as $t \to +\infty$, in general: $\int_0^t f(s)\,ds \not\to 0$ as $t \to +\infty$.

So abscissa of convergence for $\mathcal{L}[\int_0^t f]$ is usually $\geqslant 0$.

- Frequency shifts and rescalings:

$$\mathcal{L}[e^{-bt} f(t)](z) = \mathcal{L}[f](z+b)$$

$$\mathcal{L}[f(at)](z) = \frac{1}{a}\mathcal{L}[f]\left(\frac{z}{a}\right).$$

- Convolutions:

Recall: If $f, g : [0, +\infty) \longrightarrow \mathbb{C}$ are piecewise continuous,

$$f * g\,(t) = \int_{-\infty}^{+\infty} \underbrace{f(t-s)}_{=0 \text{ for } t-s<0}\, \underbrace{g(s)}_{=0 \text{ for } s<0}\, ds$$

(as if $f, g = 0$ for $t < 0$)

$$= \int_0^t f(t-s)\, g(s)\, ds.$$

Then: $\mathcal{L}[f*g] = \mathcal{L}[f] \cdot \mathcal{L}[g]$

Using the above formulas:

We can invert the Laplace transform of many explicit functions.

(We will later see a formula for the inverse of the Laplace transform in general, but it is more involved than the corresponding formula for the Fourier transform.)

Applications of the Laplace transform to the Cauchy problem (= initial value problem)

First order initial value problem:

Let $a, y_0 \in \mathbb{R}$ and let $f : [0, +\infty) \to \mathbb{R}$ a piecewise continuous function.

We want to find a function $y : [0, +\infty) \to \mathbb{R}$ such that:

$$(1) \begin{cases} y'(t) + a \cdot y(t) = f(t) & \text{for } t > 0, \\ y(0) = y_0 \end{cases}$$

Before finding the solution: We need to determine whether it has a unique solution.

We will start by showing that (1) has indeed a unique solution.

Remark: In general, linear initial value problems (i.e. not containing nonlinear terms in y, such as y^2, y^3 etc) always have a unique solution if the number of initial conditions is equal to the order of the differential equation.

In the case of (1): Let $y_1, y_2 : [0, +\infty) \to \mathbb{R}$ be two solutions. We will show that $y_1 = y_2$.

Let $w(t) = y_1(t) - y_2(t)$.

Strategy: Show that $w(t) = 0$ for all $t \geq 0$.

Observe: Since y_1, y_2 solve (1), by subtracting (1) for y_2 from (1) for y_1 (and using the fact that (1) is linear in y) we obtain:

$$\begin{cases} w'(t) + a \cdot w(t) = 0 & \text{for } t > 0, \\ w(0) = 0 \end{cases}$$

Multiply with e^{at}: $\begin{cases} \left(e^{at} w(t)\right)' = 0 \\ e^{a \cdot 0} w(0) = 0 \end{cases}$

Since $\left(e^{at} w(t)\right)' = 0 \Rightarrow e^{at} w(t) = \text{const}$

$$= e^{a0} \cdot w(0) = 0$$

So: $w(t) = 0$ for all $t \geq 0$.

So only one (aka unique) solution for (1).

Finding the solution using the Laplace transform: $y'(t) + a \cdot y(t) = f(t)$

$$\xrightarrow{\text{Laplace}} \mathcal{L}[y'](z) + a\,\mathcal{L}[y](z) = \mathcal{L}[f](z).$$

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Setting $Y(z) = \mathcal{L}[y](z)$, $F(z) = \mathcal{L}[f](z)$
and using the identity for the derivative:

① $\rightarrow$ $z \cdot Y(z) - \underbrace{y(0)}_{= y_0} + a \cdot Y(z) = F(z)$

$$\Rightarrow Y(z) = \frac{F(z)}{z+a} + \frac{y_0}{z+a}$$

The right hand side is the Laplace transform of what?

Recall: $\frac{1}{z+a} = \mathcal{L}[e^{-at}](z)$

So: $Y(z) = \mathcal{L}[e^{-at}](z) \cdot \mathcal{L}[f](z) + y_0 \cdot \mathcal{L}[e^{-at}](z)$

From formula for convolutions:

$$y(t) = \int_0^t e^{-a(t-s)} \cdot f(s)\, ds + y_0 \cdot e^{-at}$$

Let us verify that this is indeed the solution:

- $y(0) = 0 + y_0 = y_0$

- $$y'(t) = \left(\int_0^t e^{-a(t-s)} f(s)\, ds \right)' + y_0 \cdot (-a e^{-at})$$

$$= e^{-a(t-s)} f(s) \Big|_{s=t} + \int_0^t (-a) e^{-a(t-s)} f(s)\, ds - a y_0 e^{-at}$$

$$= f(t) - a \cdot \Big[\underbrace{\int_0^t e^{-a(t-s)} f(s)\, ds + y_0 e^{-at}}_{y(t)} \Big]$$

Applications to 2^{nd} order differential equations:

Consider the following initial value problem: For $w \in \mathbb{R}^*$, $a > 0$:

$$(2) \begin{cases} y''(t) + 2a\,y'(t) + w^2\,y(t) = f(t) & \text{for } t > 0 \\ y(0) = y_0 \\ y'(0) = V_0 \end{cases}$$

(2^{nd} order equation, so we need to specify 2 initial - or boundary - conditions to get a unique solution).

Remark: The above is the equation satisfied by the damped harmonic oscillator

- $f(t)$: Forcing
- $a > 0$: Damping parameter (friction)

Uniqueness of solutions: Let $y_1, y_2 : [0, +\infty) \to \mathbb{R}$ be two solutions of (2). Let $w(t) = y_1(t) - y_2(t)$.

As before, we have to show that $w(t) = 0$ for all $t \geq 0$.

Subtracting (2) for y_1 from the same for y_2:

$$\begin{cases} w''(t) + 2a\,w'(t) + \omega^2 w(t) = 0 \\ w(0) = 0 \\ w'(0) = 0 \end{cases}$$

Multiply the diff. equation with w'

$$w'' \cdot w' + 2a(w')^2 + \omega^2 \cdot w \cdot w' = 0$$

Note: $w'' \cdot w' = \frac{1}{2}\left((w')^2\right)'$, $w \cdot w' = \frac{1}{2}(w^2)'$

So:

$$\underbrace{\left(\frac{1}{2}(w')^2\right)'}_{\text{Kinetic energy}} + 2a(w')^2 + \underbrace{\left(\frac{1}{2}\omega^2 w^2\right)'}_{\text{Potential energy}} = 0$$

$$\Rightarrow \frac{d}{dt}\left(\frac{1}{2}(w')^2 + \frac{1}{2}\omega^2 w^2\right) = -2a(w')^2 \leq 0$$

So the energy $E(t) = \frac{1}{2}(w')^2 + \frac{1}{2}\omega^2 w^2$ is non-increasing (that's the effect of damping).

And:
- $E(0) = 0$ (since $w(0) = 0$, $w'(0) = 0$)
- $E(t) \geq 0$ at all times (sum of squares)

$$\Longrightarrow E(t) = 0 \text{ for all } t \geq 0$$

$$\Longrightarrow w(t) = 0 \text{ for all } t \geq 0$$

$$\left(\text{Since} \quad E(t) = \frac{1}{2}(w')^2 + \frac{1}{2}\omega^2 w^2 \geqslant \frac{1}{2}\omega^2 w^2 \right)$$

So: Unique solution

Find the solution using the Laplace transform

$$y''(t) + 2a\, y'(t) + w^2 y(t) = f(t).$$

Apply a Laplace transform:

$$z^2 Y(z) - z\, \underbrace{y(0)}_{y_0} - \underbrace{y'(0)}_{V_0} + 2a\left(z \cdot Y(z) - y_0 \right) + w^2 Y(z) = F(z)$$

Rearranging:

$$\left(z^2 + 2az + w^2 \right) Y(z) = F(z) + (z+a)\cdot y_0 + a y_0 + V_0$$

$z^2 + 2az + w^2$
$= (z+a)^2 + w^2 - a^2$

$\Longrightarrow$

(by "completing the square")

$$Y(z) = \frac{1}{(z+a)^2 + w^2 - a^2} \cdot F(z) + \frac{z+a}{(z+a)^2 + w^2 - a^2} \cdot y_0 + \frac{1}{(z+a)^2 + w^2 - a^2} \cdot (ay_0 + v_0).$$

We will consider three cases, depending on the sign of $w^2 - a^2$.

I) $a^2 = w^2$ (critically damped)

In this case: $$\frac{1}{(z+a)^2 + w^2 - a^2} = \frac{1}{(z+a)^2}$$

$$\frac{z+a}{(z+a)^2 + w^2 - a^2} = \frac{1}{z+a}$$

Recall: $\mathcal{L}[e^{-at}](z) = \dfrac{1}{z+a}$

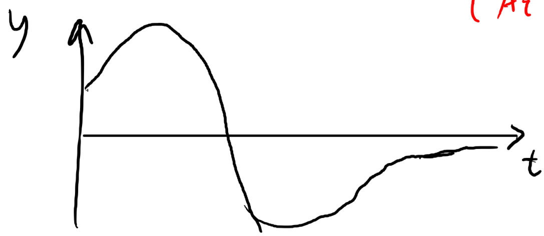
$$\Rightarrow \mathcal{L}[t \cdot e^{-at}](z) = -\frac{d}{dz}\left(\frac{1}{z+a}\right) = \frac{1}{(z+a)^2}$$

So: $$Y(z) = \mathcal{L}[t \cdot e^{-at}](z) \cdot \mathcal{L}[f](z)$$
$$+ \mathcal{L}[e^{-at}](z) \cdot y_0$$
$$+ \mathcal{L}[t \cdot e^{-at}](z) \cdot (ay_0 + v_0)$$

$$\Longrightarrow \quad y(t) = \int_0^t (t-s) \cdot e^{-a(t-s)} f(s)\, ds$$
$$+ e^{-at} \cdot y_0 + t \cdot e^{-at} \cdot (ay_0 + v_0)$$

In the case of no forcing $(f=0)$:

Typical graph of $y(t)$



(At most one root)

II) $\omega^2 > a^2$ (under-damped)

We start from the fact that

$$\mathcal{L}[\cos(\beta t)](z) = \frac{z}{z^2+\beta^2}, \quad \mathcal{L}[\sin(\beta t)](z) = \frac{\beta}{z^2+\beta^2}$$

(see Exercises - Check that the above is consistent with the formula $\mathcal{L}[e^{-i\beta t}] = \dfrac{1}{z+i\beta}$)

Then: For $\beta = \sqrt{\underbrace{\omega^2 - a^2}_{>0}}$:

$$\frac{1}{(z+a)^2 + \underbrace{\omega^2 - a^2}_{\beta^2}} = \frac{1}{\beta} \cdot \frac{\beta}{(z+a)^2 + \beta^2} = \frac{1}{\beta} \cdot \mathcal{L}\left[e^{-at} \cdot \sin(\beta t)\right](z)$$

and $$\frac{z+a}{(z+a)^2 + \underbrace{\omega^2 - a^2}_{\beta^2}} = \mathcal{L}\left[e^{-at} \cdot \cos(\beta t)\right](z)$$

So our expression becomes

$$Y(z) = \mathcal{L}\left[\frac{1}{\sqrt{\omega^2 - a^2}} e^{-at} \cdot \sin\left(\sqrt{\omega^2 - a^2}\, t\right)\right](z) \cdot F(z)$$

$$+ \mathcal{L}\left[e^{-at} \cos\left(\sqrt{\omega^2 - a^2}\, t\right)\right](z) \cdot y_0$$

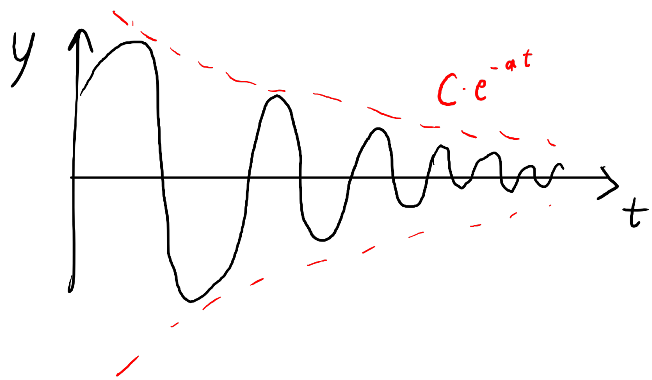
$$+ \mathcal{L}\left[\frac{1}{\sqrt{\omega^2 - a^2}} e^{-at} \cdot \sin\left(\sqrt{\omega^2 - a^2}\, t\right)\right](z) \cdot (a y_0 + v_0)$$

So $y(t) = \int_0^t \frac{1}{\sqrt{\omega^2 - a^2}} \cdot e^{-a(t-s)} \cdot \sin\left(\sqrt{\omega^2 - a^2}(t-s)\right) \cdot f(s)\, ds$

$$+ e^{-at} \cdot \cos\left(\sqrt{\omega^2 - a^2}\, t\right) \cdot y_0$$

$$+ \frac{1}{\sqrt{\omega^2 - a^2}} e^{-at} \cdot \sin\left(\sqrt{\omega^2 - a^2}\, t\right) \cdot (a y_0 + v_0)$$

When $f = 0$:



III) $\omega^2 < a^2$ (over-damped case)

In this case:

Use the fact that

$$\mathcal{L}[\cosh(\beta t)](z) = \frac{z}{z^2-\beta^2}, \quad \mathcal{L}[\sinh(\beta t)](z) = \frac{\beta}{z^2-\beta^2}$$

(Exercise, but can also be derived from the previous formula using the fact that $\cosh(x) = \cos(ix)$, $\sinh(x) = -i\sin(ix)$)

So: If $\beta = \underbrace{\sqrt{a^2-\omega^2}}_{>0}$, then:

$$\frac{1}{(z+a)^2 + \underbrace{\omega^2 - a^2}_{=-\beta^2}} = \frac{1}{\beta}\cdot\frac{\beta}{(z+a)^2-\beta^2} = \frac{1}{\beta}\mathcal{L}[e^{-at}\cdot\sinh(\beta t)](z)$$

and

$$\frac{z+a}{(z+a)^2 + \underbrace{\omega^2 - a^2}_{=-\beta^2}} = \mathcal{L}[e^{-at}\cdot\cosh(\beta t)](z)$$

So, from our previous expression:

$$Y(z) = \mathcal{L}\left[\frac{1}{\sqrt{a^2-\omega^2}} \cdot e^{-at} \sinh(\sqrt{a^2-\omega^2}\,t)\right](z) \cdot F(z)$$

$$+ \mathcal{L}\left[e^{-at} \cdot \cosh(\sqrt{a^2-\omega^2}\,t)\right](z) \cdot y_0$$

$$+ \mathcal{L}\left[\frac{1}{\sqrt{a^2-\omega^2}} e^{-at} \cdot \sinh(\sqrt{a^2-\omega^2}\,t)\right](z) \cdot (ay_0+v_0)$$

So:

$$y(t) = \int_0^t \frac{1}{\sqrt{a^2-\omega^2}} e^{-a(t-s)} \sinh(\sqrt{a^2-\omega^2}(t-s))\, f(s)\, ds$$

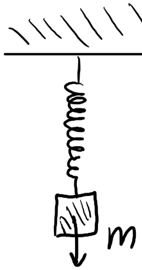
$$+ e^{-at} \cdot \cosh(\sqrt{a^2-\omega^2}\,t) \cdot y_0$$

$$+ \frac{1}{\sqrt{a^2-\omega^2}} e^{-at} \cdot \sinh(\sqrt{a^2-\omega^2}\,t) \cdot (ay_0+v_0)$$

When $f=0$:

(similar to the critically damped).

Physical motivation



Hooke's law:

air resistance

$$m \cdot y'' = f - k \cdot y - d \cdot y'$$

So $w^2 = \frac{k}{m}$, $\quad 2a = \frac{d}{m}$

Resonance: In the case when $a = 0$ (no damping), we have (when $y_0 = 0$, $v_0 = 0$):

$$y(t) = \int_0^t \frac{1}{|w|} \sin\big(|w|(t-s)\big) \cdot f(s)\, ds$$

If $f(s)$ is of the form $\sin(\gamma s)$:

- If $\gamma \neq \pm w$: $y(t)$ remains bounded as $t \to +\infty$

  ("cancellations" inside the integral)

- If $\gamma = \pm w$: $|y(t)| \longrightarrow +\infty$ as $t \to +\infty$

  (The forcing frequency = eigen-frequency)